Comparing two electrical impedance tomography algorithms: Gauss-Newton and Topology Optimization


Mechatronics Dept. Univ. of São Paulo, Av. Prof. Mello Morais 2231, São Paulo, Brazil; Mechanical Eng. Dept. Univ. of São Paulo, Av. Prof. Mello Morais 2231, São Paulo, Brazil

ABSTRACT

Electrical Impedance Tomography (EIT) seeks to recover the impedance distribution within a body using boundary data. More specifically, given the measured potentials, the model of the body - an elliptic partial differential equation - and the boundary conditions, this technique solves a non-linear inverse problem for the unknown impedance. In this work, an algorithm called Topology Optimization Method (TOM) is applied to EIT and compared to the Gauss-Newton Method. The Topology Optimization has solved some non-linear inverse problems and some of its procedures were not investigated for EIT, for instance, the use of Sequential Linear Programming. Assuming a pure resistive medium, the static resistivity distribution of a phantom was estimated using a 2-D finite element model, by two different iterative algorithms: one based on Newton-Raphson scheme and the other on Topology Optimization Method. While the first method essentially solves several algebraic systems, the second solves several linear programming problems. Results using experimental data are shown and the quality of the images obtained, time and memory used are compared for both algorithms. We intend to use these methods, in future works, for the visualization of a human lung subjected to mechanical ventilation.

Keywords: 2-D electrical impedance imaging, topology optimization, Gauss-Newton method, finite element method, absolute resistivity

1. INTRODUCTION

Electrical Impedance Tomography (EIT) allows us to obtain internal images of a body (domain). Electrodes are fixed on the boundary of the body and some low intensity electric currents, following a sinusoidal pattern, are applied through them. The resulting electric potentials are measured and, considering this data, the unknown electric impedance (resistivity and permittivity) distribution within the body is obtained by the EIT algorithm. This impedance distribution is represented by an image and constitutes the solution of a non-linear and ill-posed inverse problem. A current pattern defines the location of the current-carrying electrodes and the current intensity on each current-carrying electrode. Several current patterns are applied and therefore many electric potential values are available for image reconstruction.

The Topological Optimization Method has successfully solved some non-linear inverse problems. Some procedures of TOM have not been investigated in Electrical Impedance Tomography, namely, the use of sequential linear programming and the change of variables that in the TOM literature is often called material model. The objective of this work is to evaluate whether these procedures have advantages for EIT algorithms.

In this work we compared two EIT algorithms. The first is based on Gauss-Newton Method (GNM), a simplification of Newton-Raphson Method, and the second is based on the Topology Optimization Method (TOM). The Finite Element Method (FEM) is used to model the domain, which is a cylindrical vessel with a glass object immersed on saline. The complete electrode model is also considered and thus the impedances arising from the contact between electrodes and saline are taken into account. Both methods minimize a performance index that contains difference between electric potential measurements and computations. TOM minimizes the performance index using Sequential Linear Programming (LP) problems. On the other hand, GNM

Further author information: (Send correspondence to L. A. Motta-Mello)
L. A. Motta-Mello: E-mail: luis.mello@poli.usp.br, Telephone: 55 11 30919851
minimizes the performance index using a generalized Gauss-Newton iterative update rule, which requires some matrix inversions. Both methods look for static images. Image quality, convergence rate, amount of memory used and computing time are compared.

2. FINITE ELEMENT MODEL

In EIT, the Maxwell’s equations must be used to describe the current density flow within the body. However, these equations can be simplified to the following elliptic partial differential equation:

$$\nabla \cdot (\sigma \nabla V) = 0, \text{ in } \Omega$$

(1)

where $\sigma$ is the electric conductivity, $V$ is the scalar electric potential and $\Omega$ is the conductive body. The implemented algorithms depend on the solution of eq. 1 for the electric potentials, also known as forward problem. This solution, or the forward solution, can be uniquely determined with the knowledge of Neumann and Dirichlet boundary conditions on $\partial \Omega$; however, it cannot be obtained analytically for arbitrary geometry, conductivity distributions or boundary data. Thus, FEM was employed.

Following this method, the body is conveniently divided into finite elements – in this work, three node triangular elements – and the electric potential is approximated within each element by a known function which depends on the electric potential values on nodes of the element, the new unknowns. The electric potential is described by a finite dimensional space and the problem of finding the nodal electric potentials, $V$, turns into an algebraic problem or the following linear system of equations:

$$[K] \{V\} = \{I\}$$

(2)

where $\{V\}$ is the vector of nodal electric potentials, $\{I\}$ is vector of nodal electric currents and $[K]$ is the conductivity matrix of the system. A detailed reference in FEM can be found in the literature. When the electrodes are added to the model, the effect of the contact impedances must be taken into account. In this work, we considered the complete electrode model to represent these effects. This model gives rise to FEM matrices $[K_{el}]$ and vectors $\{I_{el}\}$ corresponding to each electrode attached to the body, which are assembled on $[K]$ and $\{I\}$, respectively. Each matrix is given by:

$$[K_{el}] = \frac{a_{el}\sigma_{el}}{t_{el}} \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1/2 & -1/2 \\ -1/2 & -1 & -1/2 & 2 \end{bmatrix}$$

(3)

and each vector by:

$$\{I_{el}\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

(4)

where $a_{el}$ is the width and $t_{el}$ is the thickness (both assumed constant) of each quadrilateral finite element which comprises the model, $\sigma_{el}$ is the conductivity of the interface and $I$ is the electric current applied to the electrode (if it is the case). $\sigma_{el}/t_{el}$ is called electrode parameter, an unknown value in practice. The fourth node in the vector corresponds to the electrode and the others must be connected to nodes of the body.

Assembling the matrices and vectors on the linear system (eq.2), we obtain the following system:

$$[K_T]\{V_T\} = \{I_T\}$$

(5)

Once the electric potential of a reference node is chosen, the system can be solved for the vector $\{V_T\}$, a non-linear function of the resistivity distribution.
3. INVERSE PROBLEM

The inverse problem is the problem solved for the electric conductivities or resistivities, given the electric currents, electric potentials on the boundary, the structure of the model, and the geometry of the domain. Mathematically, the problem can be stated as:

\[ F = \frac{1}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_p} (\Delta \phi_{ij} - \Delta \phi_{ij0})^2 \]  

(6)

where \( n_e \) is the number of current load cases, \( n_p \) is the number of measurement points, \( \Delta \phi_{ij0} \) is the relative potential obtained from the \( i \)-th pair of electrodes in the \( j \)-th current load case and \( \Delta \phi_{ij} \) is the corresponding relative potential obtained from \( \{V_T\} \), e.g., through the solution of the FEM linear system, where the relative potential is the difference between the potential of one electrode and the potential of the neighboring electrode, skipping one electrode (see Fig. 1). The minimization of this equation gives the best estimate for the conductivities and therefore for the image.

![Figure 1. The measurement of relative potential.](image)

This problem is ill-posed, which means that the best estimate for the image, based on eq. 6, may not be the best image. Therefore, regularization techniques must be applied. Each method studied, TOM and GNM, corresponds to a different regularization approach. We discuss these regularization approaches in the next subsection, in conjunction with characteristics of each method.

3.1. Topology Optimization Method

From the point of view of Bendse and Sigmund,\( ^4 \) TOM distributes material properties inside a fixed domain with the purpose of optimizing the allowed performance of the corresponding system represented (the allowed performance), in this case, by eq. 6 and some constraints or regularizations. In this work, the material properties in each point vary, continuously, from the properties of a material A to a material B based on SIMP, or Simple Isotropic Material with Penalization. The mathematical form of this statement is given by:

\[ \sigma = \rho^p \sigma_A + (1 - \rho^p) \sigma_B, \text{ in } \Omega \]  

(7)

where \( \sigma_A \) and \( \sigma_B \) are the maximum and minimum allowed conductivities, respectively, and \( \rho \) is the normalized design variable, which defines \( \sigma \) and varies from 0 to 1. Besides, when \( p \) is bigger than 1, penalization is introduced into the model, which means that intermediate values of \( \rho \) can be reduced on the final result. We used \( p=1 \) to match the GNM algorithm.

We assumed in this work that the design variables are constant in each finite element. Thus, the conductivities were given by:

\[ \sigma_m = \rho_m \sigma_A + (1 - \rho_m) \sigma_B \]  

(8)

where the subscript \( m \) indicates a finite element.

Finally, the regularized problem solved for the conductivity distribution was:

\[
\begin{align*}
\text{minimize} & \quad F = \frac{1}{2} \sum_{i=1}^{n_e} \sum_{j=1}^{n_p} (\Delta \phi_{ij} - \Delta \phi_{ij0})^2 \\
\text{w.r.t} & \quad \rho_m \\
\text{subjected to} & \quad 0 \leq \rho_m \leq 1
\end{align*}
\]  

(9)
where $M$ is the number of design variables.

This proposed non-linear problem was solved by a Sequential Linear Programming (SLP) algorithm, which solves a sequence of linear problems through Linear Programming (LP). In each LP problem (iteration), the non-linear problem above stated is linearized and the design variables are constrained between additional adjustable lateral constraints, or moving limits, which validates the linearization. A simplex based method solved each LP problem.

Besides the mentioned constraints (box constraints), we considered another regularization scheme: the spatial filter (SF), which averages the moving limits based on limits of neighboring elements, weighted by the distances from the considered element (distance between centroids). The filter radius specified defines how many elements are taken into account in the averaging process (see Fig. 2). The filter equation is given by:

\[
ml_m = \frac{\sum_q ml_q W_{mq}}{\sum_q W_{mq}} \quad (10)
\]

and

\[
W_{mq} = \frac{FR - \text{dist}_{mq}}{FR} V_q \quad (11)
\]

where $ml_m$ is the $m$-th moving limit, $FR$ is the specified filter radius, $\text{dist}_{mq}$ is the distance between $m$-th element and $q$-th element and $V_q$ is the volume of the $q$-th element. The SF is applied at each iteration and, mathematically, drives the optimization into regions in the space of feasible solutions where the distribution of the design variables is smooth.

**Figure 2.** A piece of a two-dimensional FEM mesh showing some elements within the filter radius FR, which are therefore considered in the filtering process.

Finally, the derivatives of $F$ with respect to the design variables $\rho_m$ which are used in the linearization of the problem were obtained through the adjoint method.

### 3.2. Gauss-Newton Method

The second algorithm addressed in the present work is the Gauss-Newton Algorithm in the generalized form. The update of the vector of unknowns, $\theta$, follows eq. 12

\[
\theta_{k+1} = \theta_k + \alpha (J_k^T J_k + \lambda W)^{-1} (J_k^T W (z - h(\theta_k))) - \lambda W \theta_k \quad (12)
\]

where $\theta$ denotes the vector of the unknown physical property, for instance, resistivity or conductivity, $\alpha$ is the step size, $\lambda$ is a regularization parameter, $J_k$ is the Jacobian matrix and each element of this matrix is $J_{k,i,j} = \partial (z_i - h_i(\theta))/\partial \theta_j$, $z$ is the vector of measured electrical potentials, $W$ is a regularization matrix and $h(\theta_k)$ is the observation model derived from a finite elements model.
The regularization matrix $W = L^t L$, where $L$ is a high pass filter, penalizes the high frequency content on the $\theta$ vector. The high pass filter is computed using a low pass gaussian filter, $G$, according to $L = I - G$, where $I$ is the identity matrix.

Since the Topology Optimization Method uses a change of variable according to eq. 7 with much success, in this work, two different implementations of the Gauss-Newton Algorithm are performed. One of them uses conductivity as unknowns and consequently $p = 1$ on eq. 13. The other implementation uses resistivity as unknowns and consequently $p = 1$ on eq. 13.

$$\sigma_i = \theta_i^p$$

where $p$ belongs to the set of real numbers, $\sigma_i$ is conductivity of the $i$-th element and $\theta_i$ is the $i$-th element of the vector of unknowns.

4. RESULTS

In this section, images and other results are presented. Examples using experimental data are shown. These data were obtained from a cylindrical container with 30 bar electrodes attached to its boundary (see the sketch in Fig. 3) with 35 mm high and 10mm wide (the thickness of each electrode is not necessary for the algorithm). They are equally spaced along the container, which was filled up to 35 mm with a 0.3 g/L saline solution (NaCl). Its resistivity is approximately 17 $\Omega$m. The inner diameter of the container is 300 mm. This diameter limits the conductive body. The small circle shown in Fig. 3 represents a glass object with diameter and resistivity equal to 32 mm and approximately $10^6$ $\Omega$m, respectively, which was immersed into the container. The minimum distance between the center of the object and the boundary is 30 mm.

To obtain the electrical potentials in the electrodes, a pair of them was electrically excited following the load pattern seen in Fig. 4 (this figure corresponds to one current load case). Then, the relative potentials were measured, except for those which share current carrying electrodes, due to hardware limitations. The pair of excited electrodes was successively changed until a satisfactory number of observations (or until enough information as potential values) under different angles was obtained, thus providing the necessary data for a high quality image. 30 current load cases were applied following this pattern.
The image reconstruction was carried out in a mesh with 2946 elements (including the elements of the electrode model) or 2886 unknowns, and 1564 nodes (see Fig. 5). The initial guess was 35 Ωm. Electrode parameters, which were previously obtained, are equal to 0.02 Ωm². All computations are carried out on a 3.6GHz PC with 1024MB of RAM.

![Image](image.png)

**Figure 5.** Mesh used for image estimation.

The Fig. 6 is the resistivity estimate obtained by TOM algorithm. The filter radius was chosen so that only the neighboring elements were considered in the averaging process. The position and size of the glass object were recovered. However the amplitudes and spatial variation are small, which is attributed to the SF. The resistivity of the saline is about 200 Ωm, which seems ten times higher than expected taking into account the amount of NaCl dissolved in water. One hundred iterations were required and the elapsed time was 1023 s.

![Image](image.png)

**Figure 6.** Image obtained by the Topology Optimization Algorithm TOM.
The Fig. 7 is the resistivity estimate obtained by Gauss-Newton algorithm when the unknowns are conductivities, $p = 1$. The position of the glass object was recovered properly. The peak resistivity of the glass object, $\rho = 112.04 \ \Omega m$, is much smaller than expected. The diameter of the object also resulted smaller than expected. The resistivity of the saline is about $\rho = 20 \ \Omega m$, which is on the order of the expected taking into account the amount of NaCl dissolved in water. The standard deviation of the gaussian filter is $s = 0.02$ and the regularization parameter $\alpha = 0.75$. Smaller values of the regularization parameter lead to meaningless images. Ten iterations where required and the elapsed time was 821 s.

![Newton Raphson](image1.jpg)

**Figure 7.** Image obtained by the Gauss-Newton Algorithm using conductivity as unknowns, $p = 1$

When the Gauss-Newton Algorithm was implemented using resistivity as unknowns, $p = -1$ the position of the object was recovered properly and the diameter of the object resulted as expected, but the resistivity of the object was obtained much smaller than expected $\rho = 62.0 \ \Omega m$, as shown on Fig. 8. The resistivity of the saline is about $\rho = 20 \ \Omega m$, which is on the order of the expected taking into account the amount of NaCl dissolved in water. The standard deviation of the gaussian filter is $s = 0.04$ and the regularization parameter $\alpha = 1.0e - 7$. Smaller values lead to meaningless images. Four iterations where required and the elapsed time was 341 s.

![Newton Raphson](image2.jpg)

**Figure 8.** Image obtained by the Gauss-Newton Algorithm using resistivity as unknowns, $p = -1$
The Fig. 9 is the resistivity estimate obtained by Total Variation (TV), which is defined in eq. 14, for a given continuous and differentiable function $\sigma$ of one variable $x$:

$$TV(\sigma(x)) = \int_\Omega |\nabla \sigma(x)|dx$$  \hspace{1cm} (14)
saline and larger computing time, the use of the Sequential Linear Programming on TOM seems to be beneficial for the quality of the image and for the memory used. Besides, some sharp restrictions, for instance, the non-negativeness of the conductivity, are implemented using Linear Programming without increase of numerical error propagation and has regularizing effects.

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